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ANTIPLANE WAVES IN AN ELASTIC MEDIUM WITH A BIPERIODIC SYSTEM OF CAVITIES†

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Using the symmetric-potential method, a numerical and qualitative investigation is made of antiplane (S–H) waves in a homogeneous isotropic elastic medium with a system of circular cylindrical cavities with group symmetry C_{2v}^2 [1]. © 1998 Elsevier Science Ltd. All rights reserved.

1. We will consider antiplane stationary wave motions in a homogeneous isotropic elastic space, where the displacements of points are parallel to the x_3 axis and the magnitude of these displacements are independent of that coordinate. In this case, the amplitude values of the displacements satisfy the Helmholtz equation

$$\Delta u + \varkappa^2 u = 0, \quad \varkappa = \omega / \upsilon \tag{1.1}$$

(ω is the angular frequency of the wave and υ is the velocity of propagation of transverse waves in the medium).

We shall seek a solution of the Helmholtz equation with a special form of right-hand side: a system of E_M Dirac delta functions concentrated at the points $x_1 = ml_1, x_2 = nl_2$ ($m, n = 0, \pm 1, \pm 2, \ldots, M$) and multiplied by $[i(m\alpha_1 + n\alpha_2)]$

$$\Delta u + \varkappa^2 u = \sum_{m,n=-M}^{M} \exp[i(m\alpha_1 + n\alpha_2)]\delta(x_1 - ml_1)\delta(x_2 - nl_2) \tag{1.2}$$

Here l_j, α_j ($j = 1, 2$) are certain constants ($|\alpha_j| \leqslant \pi$). Following the maximum absorption principle, we replace $\varkappa^2$ in (1.2) by $\varkappa^2 + i\varepsilon$, and then perform a double Fourier transformation with respect to the coordinates x_1, x_2. We obtain

$$u(\xi_1, \xi_2) = \frac{1}{\varkappa^2 + i\varepsilon - \xi_1^2 - \xi_2^2} \sum_{m,n=-M}^{M} \exp[im(l_1\xi_1 - \alpha_1) + in(l_2\xi_2 - \alpha_2)] \tag{1.3}$$

We then perform the double inverse Fourier transformation in (1.3), meaning that we take the limit [3]

$$u(x_1, x_2) = \lim_{\xi \to \infty} \iint_{\xi_1^2 + \xi_2^2 < \xi^2} u(\xi_1, \xi_2) \exp(-i\xi_1 x_1 - i\xi_2 x_2) dx_1 dx_2 \tag{1.4}$$

We distribute the system E_M over the whole plane, that is, we let M tend to infinity. Taking the limit as $M \to \infty$ in (1.3) and (1.4), using the equation [3]

$$\sum_{m=-\infty}^{\infty} \exp[-im(\alpha - \xi l)] = \frac{2\pi}{l} \sum_{k=-\infty}^{\infty} \delta\left(\xi - \frac{2k\pi - \alpha}{l}\right)$$

and putting $\varepsilon = 0$, from (1.4) we will have

$$\Gamma(\alpha, \omega, \mathbf{x}) = \frac{1}{l_1 l_2} \sum_{k,j} G_{kj} \exp(-i\xi_{1,k} x_1 - i\xi_{2,j} x_2) \tag{1.5}$$

$$G_{kj} = (\varkappa^2 - \xi_{1,k}^2 - \xi_{2,j}^2)^{-1}, \quad \xi_{1,k} = \frac{2k\pi - \alpha_1}{l_1}, \quad \xi_{2,j} = \frac{2j\pi - \alpha_2}{l_2}; \quad \alpha = (\alpha_1, \alpha_2)$$

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Similarly, the double sum in (1.5) is taken to mean the limit of the corresponding finite sum with $\xi_{1,k}^2 + 2\xi_{2,j}^2 \leqslant \xi^2$ as $\xi \to \infty$.

Note that

$$\Gamma(\alpha,\omega,x_1+nl_1,x_2+ml_2)=\exp(in\alpha_1+im\alpha_2)\Gamma(\alpha,\omega,x_1,x_2) \tag{1.6}$$

Below we will call functions such as this α-periodic, or transformed according to an irreducible representation of the subgroup of translations.

Note that as the frequency ω tends to $\xi_{kj} = (\xi_{1,k}^2 + \xi_{2,j}^2)^{1/2}$ the function Γ(α, ω, x) (we call it Green's α-periodic function) tends to infinity, indicating resonance.

We will show that the series in (1.5) converges when $\omega \neq \xi_{kj}$ and $|\mathbf{x}| \neq 0$. Using a similar method to that employed in [4], we derive the two-dimensional analogue of the Euler–Maclaurin formula

$$\begin{aligned}\sum\sum_{\rho_k,\eta_j\in\Pi} f(\rho_k,\eta_j)=&\frac{1}{h_1h_2}\iint_\Pi f(v,w)dvdw+\frac{1}{2}\left[\frac{1}{h_1}\int_{v_0}^{v_1}(f(v,w_1(v))+f(v,w_0(v)))dv+\right.\\&\left.+\frac{1}{h_2}\int_{w_0(v_0)}^{w_1(v_0)}f(v_1,w)dw+\frac{1}{h_2}\int_{w_0(v_1)}^{w_1(v_1)}f(v_1,w)dw\right]+\\&+\frac{1}{12}\left[\frac{1}{h_1}\int_{v_0}^{v_1}(f_{01}(v,w_1(v))-f_{01}(v,w_0(v)))dv+\frac{1}{h_2}\int_{w_0(v_1)}^{w_1(v_1)}f_{10}(v_1,w)dw-\right.\\&\left.-\frac{1}{h_2}\int_{w_0(v_0)}^{w_1(v_0)}f_{10}(v_0,w)dw\right]+O(\max_{v,w\in\Pi}|f(v,w)|)\end{aligned}$$

$$\rho_k=h_1k+a_1,\qquad \eta_j=h_2j+a_2;\qquad f_{mn}=\frac{\partial^{m+n}f}{\partial v^m\partial w^n} \tag{1.7}$$

Here Π is a region in the υ, w plane which possesses the following property: the intersection of any straight line υ = const with this region is either empty or a single segment; $w = w_0(\upsilon)$ and $w = w_1(\upsilon)$ are the lower and upper boundaries of Π, and υ_0 and υ_1 are the abscissae of the extreme left- and right-hand points of this region.

Note that for k and j of large modulus

$$G_{kj} \sim -(\xi_{1,k}^2+\xi_{2,j}^2)^{-1}$$

We will apply the asymptotic expansion (1.7) to formula (1.5), assuming that Π is the part of the ring $r_1^2 \leqslant \upsilon^2 + w^2 \leqslant r_2^2$ which lies in the first quadrant of the plane uw, $\rho_k = \xi_{1,k}$, $\eta_j = \xi_{2,j}$, $h_1 = 2\pi/l_1$, $h_2 = 2\pi/l_2$; $f(\upsilon, w) = -\exp(-i\upsilon x_1 - iwx_2)/(\upsilon^2 + w^2)$. Then $w_0 = (r_1^2 - \upsilon^2)^{1/2}$ when $\upsilon \leqslant r_1$ and $w_0 = 0$ for $w_1 \leqslant \upsilon \leqslant r_2$; $w_1 = (r_2^2 - \upsilon^2)^{1/2}$.

It can be seen that the single integrals on the right-hand side of (1.7) are of the order of $1/r_1$. By following a similar argument for the other quadrants of the υ, w plane and adding the resulting formulae, after transforming to polar coordinates $\upsilon = r\cos\theta$, $w = r\sin\theta$ in the double integral we find that

$$\begin{aligned}&\frac{1}{l_1l_2}\sum\sum_{r_1^2\leqslant\xi_{1,k}^2+\xi_{2,j}^2\leqslant r_2^2} G_{kj}\exp(-i\xi_{1,k}x_1-i\xi_{2,j}x_2)=\\&=-\frac{1}{4\pi^2}\int_{r_1}^{r_2}\frac{1}{r}\int_{-\pi}^{\pi}\exp[-i|x|r\sin(\theta+\phi)]d\theta dr+O\left(\frac{1}{r_1}\right)=-\frac{1}{2\pi}\int_{r_1}^{r_2}J_0(|x|r)\frac{dr}{r}+O\left(\frac{1}{r_1}\right)\end{aligned} \tag{1.8}$$

The angle φ is found from the conditions: $\sin\phi = x_1/|\mathbf{x}|$, $\cos\phi = x_2/|\mathbf{x}|$, $|\mathbf{x}| = (x_1^2 + x_2^2)^{1/2}$. The last equation in the chain (1.8) was obtained using the periodicity of the integrand with respect to θ and the Sommerfeld integral representation

$$J_n(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(iz\sin\phi-in\phi)d\phi \tag{1.9}$$

The integral

$$\int_1^{\infty} J_0(|\mathbf{x}|r)\frac{dr}{r}$$

converges for $|\mathbf{x}| \neq 0$, and so the last stage in the chain of Eqs (1.8) can be made as small as desired for sufficiently large r_1. This also implies that the double series in (1.5) is convergent when $|\mathbf{x}| \neq 0$.

We now put $r_2 = \infty$. Integrating by parts in the last stage of (1.8), we find that it is of the order of $(2\pi)^{-1}\ln|\mathbf{x}|$. This means that Green's α-periodic function $\Gamma(\alpha, \omega, x)$ has a logarithmic singularity at $|\mathbf{x}| = 0$.

Isolating this singularity, we write

$$\Gamma(\alpha,\omega,\mathbf{x}) = R(\alpha,\omega,\mathbf{x}) + (2\pi)^{-1}\ln|\mathbf{x}| \tag{1.10}$$

$$R(\alpha,\omega,\mathbf{x}) = \frac{1}{l_1 l_2}\sum_{k,j}(G_{kj} - g_{kj})\exp(-i\xi_{1,k}x_1 - i\xi_{2,j}x_2)$$

Here g_{kj} are the coefficients of the expansion of the function $(2\pi)^{-1}\ln|\mathbf{x}|$ in a double Fourier series in terms of the functions $\exp(-i\xi_{i,k} - i\xi_{2j})$ $(k, j = 0, \pm1, \pm2, \ldots)$. The series (1.11) is absolutely convergent and is a continuously differentiable function of $\mathbf{x}$.

2. Consider a homogeneous isotropic elastic medium with a system of circular cylindrical cavities which is invariant under transformations of the group C_{2v}^2 consisting of reflections in two systems of parallel planes σ_{jk} $(j = 1, 2; k = 0)$ and translations (shears) by vectors which are multiples of the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ (Fig. 1). We isolate the so-called basic rectangle, which, when acted on by transformations of the subgroup of translations, can cover the entire medium (shown by the bold continuous line). We place the origin of the complete system of coordinates x_1x_2 at the centre of the basic rectangle. Similarly, we choose a basic elementary cell, which when acted on by all transformations of the group C_{2v}^2 can cover the medium (shown by the bold dashed line). On each of the elementary cells (obtained from the basic cell using elements of the symmetry group) we select a local system of coordinates such that they transform into one another in the given transformations. The set of these systems of coordinates is called an invariant system.

Theorem. The frequencies of harmonic waves in an elastic medium with a discrete spatial symmetry group **G** split up into series corresponding to irreducible representations of that group: the amplitude functions of the displacements of points of the media corresponding to each frequency can be selected in such a way that they are transformed according to a definite irreducible representation of the group **G**.

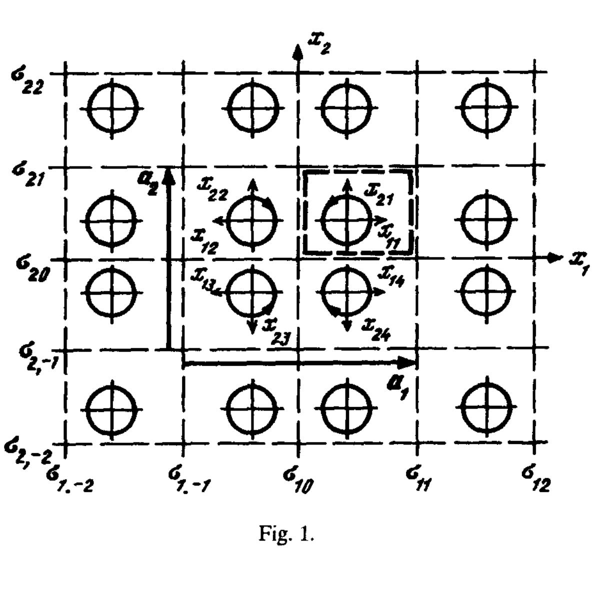

Fig. 1.

Proof. By Bloch's theorem [5], to each harmonic wave in a medium which is invariant with respect to the subgroup T of translations, there corresponds an amplitude function of displacements of the form $u(\mathbf{x}) = w(\mathbf{x}) \exp(i\mathbf{kx})$, where the function $w(\mathbf{x})$ is invariant under translation and $\mathbf{k}$ is a wave vector. The function $u(\mathbf{x})$ is transformed according to an irreducible representation of the group of translations.

In fact

$$u(\mathbf{x}+m_1\mathbf{a}_1+m_2\mathbf{a}_2)=u(\mathbf{x})\exp(im_1\alpha_1+m_2\alpha_2),\quad \alpha_j=\mathbf{ka}_j \quad (j=1,2)$$

We act on $u(\mathbf{x})$ by elements of the point subgroup $\mathbf{H} \in \mathbf{G}$

$$u_j(\mathbf{x})=g_j u(\mathbf{x})=w_j(\mathbf{x})\exp(i\mathbf{k}_j\mathbf{x}) \quad (g_j \in H)$$

($w_j(\mathbf{x}) = g_j w(\mathbf{x})$, $k_j = g_j\mathbf{k}$, $j = 1, 2, \dots, N$ and N is the order of the group H).

By the symmetry of the medium, $u_j(\mathbf{x})$ are amplitude functions of waves corresponding to one and the same frequency, and if all the k_j ($j = 1, 2, \dots, N$) are different, it follows from the way in which these functions are constructed that they transform according to an irreducible representation of the group $\mathbf{G}$ [6]. But if $\mathbf{k}_j = \mathbf{k}_i$ for some j and i (that is, $\mathbf{k}_j$ is invariant with respect to the elements of some subgroup $\mathbf{H}_0 \in \mathbf{H}$), then $u_j(\mathbf{x})$ and $u_i(\mathbf{x})$ are replaced by linear combinations of them, transformed according to irreducible representations of the group $\mathbf{H}_0$.

(A similar theorem for the free oscillations of elastic systems with point symmetry group was proved in 1970 in the author's candidate dissertation.)

The Brillouin zone for the Bravais lattice [2] with space symmetry group C_{2v}^2 has the form shown in Fig. 2. The vectors $\mathbf{b}$, which define this region, can be computed from the formula $\mathbf{b}_j = \mathbf{a}_j/(2l_j)^2$, $l_j = |\mathbf{a}_j|/2$ ($j = 1, 2$). The irreducible representations of this group depend on the vector $\mathbf{k} = \alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2$. We shall first consider the case when the star of the vector $\mathbf{k}$ consists of the four vectors $\pm\alpha_1\mathbf{b}_1 \pm\alpha_2\mathbf{b}_2$ (Fig. 2a). In this case, an irreducible representation of the group is four-dimensional and is uniquely defined by the parameter α [2].

We shall seek the amplitude displacements of points of the medium during wave motion corresponding to this irreducible representation in the form of the sum of simple layer potentials

$$u(\mathbf{x})=\sum_{l=1}^{l_0}\int_{\Omega_l}\Gamma(\alpha,\omega,\mathbf{x}-\mathbf{y}_l)q_l(\mathbf{y}_l)ds_l \quad (\mathbf{y}_l\in\Omega_l, l_0=4) \tag{2.1}$$

The directions of integration along the contours Ω_l are shown in Fig. 1. Green's function $\Gamma(\alpha, \omega, \mathbf{x})$ is calculated using formula (1.5) with the quantity l_j replaced by $2l_j$ ($j = 1, 2$).

It can be shown that the effect of elements of the group C_{2v}, consisting of the identity element, reflections in planes $\sigma_{1,0}$, $\sigma_{2,0}$ and rotation through 180° about the x_3 axis will translate the function $u(\mathbf{x})$ into the functions $u_j(\mathbf{x})$ ($j = 1, 2, 3, 4$) (where $u_1(\mathbf{x}) = u(\mathbf{x})$), transformed according to a four-dimensional irreducible representation of the group C_{2v}^2, where the only difference between the $u_j(\mathbf{x})$ lies in the numbering of the potential densities $q_l(\mathbf{y})$ ($l = 1, 2, 3, 4$). We shall call these potentials symmetric.

Suppose that on contours Ω_j ($j = 1, 2, \dots, l_0$) the boundary conditions are

$$u(\mathbf{x})=0,\quad \mathbf{x}\in\Omega_j\,(j=1,2,\dots,l_0) \tag{2.2}$$

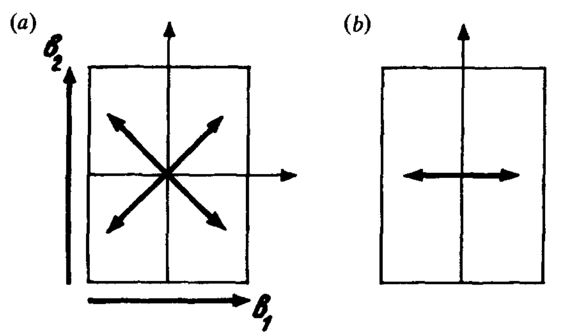

Fig. 2.

Note that it follows from the way in which the functions $u_j(\mathbf{x})$ ($j = 1, 2, 3, 4$) have been constructed that if the boundary conditions (2.2) are satisfied for one function $u_1(\mathbf{x})$ on four contours of the basic rectangle, they are also satisfied for all $u_j(\mathbf{x})$ on all contours.

Substituting expression (2.1) into (2.2), we obtain the system of integral equations

$$\sum_{l=1}^{l_0} \int_{\Omega_l} \Gamma(\alpha,\omega,\mathbf{x}_p-\mathbf{y}_l)q_l(\mathbf{y}_l)ds_l = 0 \quad (\mathbf{x}_p \in \Omega_p, \quad p=1,2,\ldots,l_0) \tag{2.3}$$

We then have

$$\Gamma(\alpha,\omega,\mathbf{x}_p-\mathbf{y}_p) = S(\mathbf{x}_p-\mathbf{y}_p)+R(\alpha,\omega,\mathbf{x}_p-\mathbf{y}_p)$$

$$S(\mathbf{x}_p) = (2\pi)^{-1}\ln(x_{1,p}^2+x_{2,p}^2)^{1/2}$$

$$R(\alpha,\omega,\mathbf{x}_p) = (4l_1l_2)^{-1}\sum_{k,j}(\Gamma_{kj}-g_{kj})\exp(-i\xi_{1,k}^{(p)}x_{1,p}-i\xi_{2,j}^{(p)}x_{2,p}) \tag{2.4}$$

$$\Gamma(\alpha,\omega,\mathbf{x}_p-\mathbf{y}_l) = (4l_1l_2)^{-1}\sum_{k,j}\Gamma_{kj}\exp\{-i\xi_{1,k}^{(p)}[x_{1,p}-P_{pl}y_{1,l}+e_1(1-P_{pl})]-$$

$$-i\xi_{2,j}^{(p)}[x_{2,p}-Q_{pl}y_{2,l}+e_2(1-Q_{pl})]\} \quad (p\neq l)$$

Here $x_{1,p}, x_{2,p}, y_{1,l}, y_{2,l}$ are the coordinates of the points $\mathbf{x}_p$ and $\mathbf{y}_l$($\mathbf{x}_p \in \Omega_p, \mathbf{y}_l \in \Omega_l$) in the corresponding local systems of coordinates, e_1, e_2 are the coordinates of the centre of the circle Ω_1 in the complete system, P_{pl} and Q_{pl} ($p, l = 1, 2, \ldots, l_0$) are the elements of symmetric matrices, where

$$P_{12}=-1,\quad P_{13}=-1,\quad P_{14}=1,\quad P_{23}=1,\quad P_{24}=-1,\quad P_{34}=-1$$

$$Q_{12}=1,\quad Q_{13}=-1,\quad Q_{14}=-1,\quad Q_{23}=-1,\quad Q_{24}=-1,\quad Q_{34}=1$$

and finally

$$\xi_{1,k}^{(p)} = [2k\pi-(-1)^{E(p/2)}\alpha_1]/(2l_1),\quad \xi_{2,j}^{(p)} = [2j\pi-(-1)^{E[(p-1)/2]}\alpha_2]/(2l_2)$$

($E(x)$ is the integer part of the number x).

It follows from (2.4) that (2.3) is a system of Fredholm integral equations of the first kind.

In each of the local systems of coordinates x_{1p}, x_{2p} we will introduce the polar system r_p, ϕ_p (or θ_p) (Fig. 1) and represent the coordinates of the points x_p and y_l in the following form

$$x_{1,p}=r_p\cos\theta_p,\quad x_{2,p}=r_p\sin\theta_p,\quad y_{1,l}=r_l\cos\phi_l,\quad y_{2,l}=r_l\sin\phi_l \tag{2.5}$$

We shall seek a solution of system (2.3) in the form of the series

$$q_l(\mathbf{y}_l) = \sum_{m=-\infty}^{\infty} q_{lm}\exp(im\phi_l) \tag{2.6}$$

We substitute (2.6) into (2.3), then multiply by $(2\pi)^{-1}\exp(-in\theta_p)$ and integrate over θ_p from $-\pi$ to $+\pi$. As a result, we obtain an infinite system of algebraic equations

$$A_{1,p,n}q_{pn}+\sum_{l=1}^{l_0}\sum_{m=-\infty}^{\infty}A_{2,p,l,n,m}q_{lm}=0 \tag{2.7}$$

$$(p=1,2,\ldots,l_0;\quad n=0,\pm1,\pm2,\ldots)$$

where

$$A_{1,p,n} = r_0\ln r_0\delta_{n,0}-\frac{r_0}{2|n|}(1-\delta_{n,0})$$

$$A_{2,p,p,n,m} = \frac{\pi r_0}{2l_1l_2}\sum_{k,j}(\Gamma_{kj}-g_{kj})F_{kjn}^{(p)}E_{k,j,-m}^{(p)}$$

$$\begin{Bmatrix} E^{(p)} \\ F^{(p)} \end{Bmatrix}_{kjn} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[\pm i r_0 U_{kj}^{(p)}(\theta) - in\theta] d\theta \tag{2.8}$$

$$U_{kj}^{(p)}(\theta) = \xi_{1,k}^{(p)} \cos\theta + \xi_{2,j}^{(p)} \sin\theta$$

Using the representation (1.9), we can show that

$$E_{kjn}^{(p)} = \exp(in\psi_{kj}^{(p)}) J_n(r_0 \xi_{kj}^{(p)}), \quad F_{kjn}^{(p)} = (-1)^n E_{kjn}^{(p)} \tag{2.9}$$

where $\psi_{kj}^{(p)}$ is determined from the conditions

$$\sin\psi_{kj}^{(p)} = \xi_{1,k}^{(p)} / \xi_{kj}^{(p)}, \quad \cos\psi_{kj}^{(p)} = \xi_{2,j}^{(p)} / \xi_{kj}^{(p)}, \quad \xi_{kj}^{(p)} = [(\xi_{1,k}^{(p)})^2 + (\xi_{2,j}^{(p)})^2]^{1/2}$$

The other coefficients of system (2.7) have similar representations

$$A_{2,p,l,n,m} = \frac{\pi r_0}{2 l_1 l_2} \sum_{k,j} \Gamma_{kj} \exp[-i\xi_{1,k}^{(p)} e_1 (1 - P_{pl}) - i\xi_{2,j}^{(p)} e_2 (1 - Q_{pl})] F_{kjn}^{(p)} E_{kj,-m}^{(l)} \quad (p \neq l) \tag{2.10}$$

Multiplying both sides of Eq. (2.7) by $| n |$ for $n \neq 0$, we obtain an infinite system of linear algebraic equations

$$B_{1,p,n} q_{pn} + \sum_{l=1}^{l_0} \sum_{m=-\infty}^{\infty} B_{2,p,l,n,m} q_{lm} = 0 \tag{2.11}$$

$$(p = 1, 2, \dots, l_0; \ n = 0, \pm 1, \pm 2, \dots)$$

$$B_{1,p,n} = r_0 \ln r_0 \delta_{n,0} - r_0 (1 - \delta_{n,0}) / 2$$

$$B_{2,p,l,0,m} = A_{2,p,l,0,m}, \quad B_{2,p,l,n,m} = A_{2,p,l,n,m} |n| \quad (n \neq 0)$$

Using the smoothness of the functions $R(\alpha, \omega, \mathbf{x}_p - \mathbf{y}_p)$ and $\Gamma(\alpha, \omega, \mathbf{x}_p - \mathbf{y}_l)$ $(p \neq l)$ for values of $\mathbf{x}_p$ and $\mathbf{y}_l$ satisfying (2.5) on the square $| \theta, \phi | \leq \pi$, we can see that $\Sigma | B_{2p,l,n,m} |^2 < \infty$ and, therefore, (2.11) is normal to a Koch system and can be solved by reduction [7].

We will consider the case where the "lattice" wave vector **k** is collinear to but not equal to half the vector $\mathbf{b}_1$ (Fig. 2b). In that case the star of the vector **k** consists of two vectors: **k** and **–k**. The point group of the vector **k** consists of two elements: the identity element and reflections in the plane $\sigma_{1,0}$ and has two one-dimensional irreducible representations with numbers $j = 1, 2$. Then the two-dimensional irreducible representations of the group C_{2v}^2 with the same numbers correspond to the vector **k** (and therefore the parameter α also). We will construct the displacement functions transformed according to these representations. In expression (2.1) we must put $q_1(\phi_1) = (-1)^j q_4(\phi_4)$, $q_2(\phi_2) = (-1)^j q_3(\phi_3)$ $(j = 1, 2)$. When comparing the system of integral equations, again, the only symmetry conditions that need to be satisfied are the boundary conditions on the contours Ω_1 and Ω_2. Thus we arrive at a system of the form (2.3) in which $l_0 = 2$. Similar changes are made to the subsequent formulae also.

Now let the star of the vector **k** consist of only one vector ($\mathbf{k} = 0$, say). Since all the irreducible representations of the group c_{2v} are one-dimensional, all the representations of the group C_{2v}^2 in this case are also one-dimensional and the representations of both groups can be given the same numbers. In the formulae given above, $l_0 = 1$.

Substituting the solution of the truncated system (2.11) into (2.1), we arrive at the expression

$$u(\mathbf{x}) = u_0(\mathbf{x}) \exp[i\alpha_1 x_1/(2l_1) + i\alpha_2 x_2/(2l_2)] \tag{2.12}$$

$$u_0(\mathbf{x}) = \frac{r_0}{4 l_1 l_2} \sum_{k,j} \Gamma_{kj} \exp(-is_{1,k} - is_{2,j}) \sum_{l=1}^{l_0} \sum_{m=-n_0}^{n_0} q_{lm} H_{kjlm}$$

$$H_{kjlm} = \int_{-\pi}^{\pi} \exp(i\xi_{1,k}^{(l)} y_{1,l} + \xi_{2,j}^{(l)} y_{2,l} + im\theta_l) d\theta_l; \quad s_{1,k} = \frac{k\pi}{l_1}, \quad s_{2,j} = \frac{j\pi}{l_2}$$

Note that the function $u_0(\mathbf{x})$ in representation (2.12) is invariant under subgroup of translations, and the exponential multiplier is transformed according to an irreducible representation of this sub-group.

Allowing for the fact that the displacements depend on time we can write

$$u(\mathbf{x}, t) = |u_0(\mathbf{x})| \exp\{i[a_0(\mathbf{x}) + k_1x_1 + k_2x_2 - \omega t]\}, \quad a_0(\mathbf{x}) = \arg u_0(\mathbf{x})$$

This shows that on a biperiodic lattice $\mathbf{x}_{mn} = \mathbf{x}_{00} + m\mathbf{a}_1 + n\mathbf{a}_2$ ($\mathbf{x}_{00}$ is an arbitrary point, $m, n = 0, \pm1, \pm2, \ldots$), $u(\mathbf{x}, t)$ behaves like a plane wave, since $|u_0(\mathbf{x})|$ and $a_0(\mathbf{x})$ have the same values at every lattice point. We shall refer to these as lattice-plane waves (LP-waves). The vector $\mathbf{k}$ plays the part of a wave vector which assigns the direction of motion of the wave. The wavelength $\lambda = 2\pi/|\mathbf{k}| = 2\pi/(\alpha_1^2/l_1^2 + \alpha_2^2/l_2^2)^{1/2}$ and the phase velocity $\upsilon = \omega/|\mathbf{k}| = \omega/(\alpha_1^2/l_1^2 + \alpha_2^2/l_2^2)^{1/2}$.

Generally speaking, of course, an LP-wave is not a plane wave. The biperiodic function $|u_0(\mathbf{x})|$ defines the "amplitude modulation" of the LP-wave, and $a_0(\mathbf{x})$ defines the "phase modulation". $|u_0(\mathbf{x})|$ is called the waveform.

3. The determinant of the truncated system (2.7) (and therefore (2.11)) takes real values.

For, from (2.10) we have

$$\bar{A}_{2,p,l,n,m} = \frac{r_0}{2\pi} \int_{-\pi}^{\pi}\int_{\pi}^{\pi} \bar{\Gamma}(\alpha, \omega, \mathbf{x}_p - \mathbf{y}_l) \exp(-im\phi_l + in\theta_p) d\phi_l d\theta_p$$

It follows from (1.5) that $\Gamma^-(\alpha, \omega, \mathbf{x}_p - \mathbf{y}_l) = \Gamma(-\alpha, \omega, \mathbf{x}_p - \mathbf{y}_l)$. A change in the parameter α of this kind corresponds to a transition from one local system of coordinates to another. Thus $\Gamma^-(\alpha, \omega, \mathbf{x}_p - \mathbf{y}_l) = \Gamma(-\alpha, \omega, \mathbf{x}_{h(p)} - \mathbf{y}_{h(l)})$, where $h(p)$ is a permutation of the symbols $1, 2, \ldots, l_0$. Hence

$$\bar{A}_{2,p,l,n,m} = A_{2,h(p),h(l),-n,-m} \tag{3.1}$$

It follows from (3.1) that performing the complex conjugate operation on the determinant of the matrix of the truncated system (2.7) $\Delta_j(\alpha, \omega)$ (the parameters α and j define an irreducible representation of the group C_{2v}^2) will lead to an identical permutation of rows and columns of the matrices, which leaves the determinant unchanged: $\Delta_j^-(\alpha, \omega) = \Delta_j(\alpha, \omega)$, that is, $\Delta_j(\alpha, \omega)$ is a real quantity.

The roots of the determinant $\omega_k(\alpha_1, \alpha_2)$ ($k = 1, 2, \ldots$) form surfaces in the space of the three variables $\omega(\alpha_1, \alpha_2)$. We shall call them frequency surfaces (FS). As noted above, if in the given perforated medium an LP-wave propagates with frequency ω and wave vector $\mathbf{k}$, an LP-wave with the same frequency and with wave vector $\tilde{\mathbf{k}}$, where $\tilde{\mathbf{k}} = (-k_1, k_2)$ or $\mathbf{k} = (k_1, -k_2)$, can also propagate in it. Thus the LP are symmetric about the planes $\alpha_1 = 0$ and $\alpha_2 = 0$. On the other hand, if α is increased by 2π, $\xi_{i,k}$ will become equal to $\xi_{i,k} = [(k-1)2\pi - \alpha_2]/l$. Replacing k by $k + 1$ in (1.5), we obtain $\Gamma = (\tilde{\alpha}, \omega, \mathbf{x}) = \Gamma(\alpha, \omega, \mathbf{x})$, $\alpha = (\alpha_1 + 2\pi, \alpha_2)$ [or $\tilde{\alpha} = (\alpha_1, \alpha_2 + 2\pi)$], whence it follows that the LP are biperiodic with respect to α_1 and α_2 with identical periods 2π. Thus, it is sufficient to define the LP on the square $0 \leqslant \alpha_i \leqslant \pi$ ($i = 1, 2$).

4. But if the holes have free contours, by an argument similar to that used in Section 2 we obtain the expression

$$\tau(\mathbf{x})/(2\mu) = \sum_{l=1}^{l_0} \int_{\Omega_l} \Gamma^{(n)}(\alpha, \omega, \mathbf{x} - \mathbf{y}_l) q_l(\mathbf{y}_l) ds_l \tag{4.1}$$

Here $\tau(\mathbf{x})$ are shear stresses on the area with normal n situated at the point $\mathbf{x}$; and $\Gamma^{(n)}(\mathbf{x})$ is half the derivative of the α-periodic Green's function in the direction of that normal.

Taking the limit as $\mathbf{x} \to \mathbf{x}_p$ ($\mathbf{x}_p \in \Omega_p$) in (4.1), we obtain a system of Fredholm integral equations of the second kind which converge to system (2.11), in which

$$B_{1,p,n} = \frac{1}{4}(\delta_{n,0} + 1)$$

$$B_{2,p,l,n,m} = \frac{r_0}{2\pi} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Gamma^{(n)}(\alpha, \omega, \mathbf{x}_p - \mathbf{y}_l) \exp(im\phi_l - in\theta_p) d\phi_l d\theta_p$$

As in Section 3, $B_{2,p,l,n,m}$ can be expressed in terms of Bessel functions of the first kind. As in the case of fixed holes, we have a normal Koch system which can be solved by reduction.

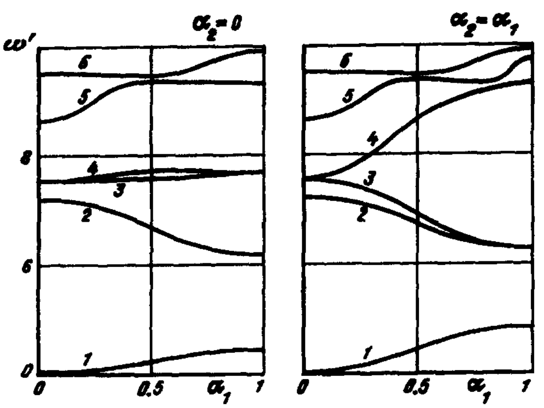

Fig. 3.

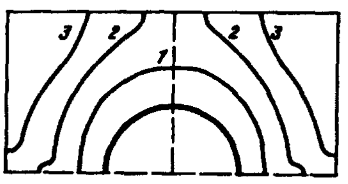

Fig. 4.

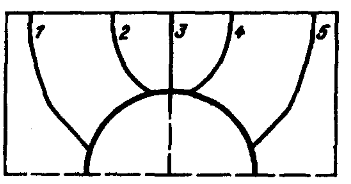

Fig. 5.

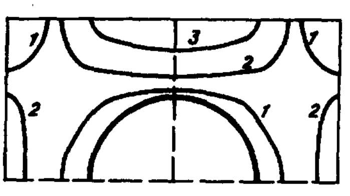

Fig. 6.

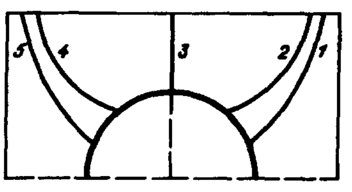

Fig. 7.

5. *Examples.* We will consider an example in which $e_i = l_i/2$ (e_i are the coordinates of the centre of the hole in the basic unit cell, $i = 1, 2$) and we will take the dimensionless variables: $x'_i = x_i/l_1$ ($i = 1, 2$), $\omega' = \omega l_1/\upsilon$. We will investigate the case where the contours of the holes are fixed. We take $l'_1 = l'_2 = 1$, $r'_0 = 0.25$. For the given values of α_1 and α_2, in the series (2.6) we keep $2n_0 + 1$ terms from $-n_0$ to $+n_0$, thereby truncating the infinite system (2.11). To find the roots ω'_k of the determinant of this system for each value of ω' the quantity n_0 is chosen so that a further increase results in a fairly small change in the determinant (for $n_0 \geqslant 5$ the determinant was computed to accuracy 10^{-4} when constructing the graphs).

Figure 3 shows sections of surfaces by the planes $\alpha_2 = 0$ and $\alpha_2 = \alpha_1$ (when $l'_1 = l'_2$ the LP are still symmetric about the planes $\alpha_2 = \pm\alpha_1$). The surfaces are numbered in order of increasing frequencies.

Figure 4 gives the level lines of the waveform of LP 1 for $\alpha_1 = \pi/2$, $\alpha_2 = 0$. The numbers 1, 2 and 3 correspond to levels 0.25, 0.5 and 0.75 (the function $|u_0(\mathbf{x})|$ is normalized so that $\max u_0(\mathbf{x}) = 1$; only half the picture is shown, as it is symmetric).

Figure 5 shows how the zero line (line $\operatorname{Re} u(\mathbf{x}, t) = 0$) moves, that is, its positions at $t = (j-1)T/8$ ($i = 1, 2, 3, 4$), j is the corresponding number in Fig. 4; $T = 2\pi/\omega$ on the "basic" square $|x'_i| \leqslant l'_i/2$ ($i = 1, 2$). When $t = T/2$ there are two zero lines on the square (denoted by the numbers 1 and 5). During the next half-period the line occupies positions 2, 3 and 4 at times $t = (j-1)T/8 + T/2$ ($j = 2, 3, 4$), but the displacements to the left and right of the line have opposite signs to those for the first half-period. We see that the waves move to the right, the direction of the lattice wave vector $\mathbf{k}$.

The level lines of the waveform for LP2 are shown in Fig. 6, and the motion of the waves is shown in Fig. 7; in this case, as we see, the wavefronts move to the left, in the opposite direction to $\mathbf{k}$.

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